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Adaptive Control Design for Nonaffine Models Arising in Flight Control

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Adaptive tracking control algorithms are developed for a class of models encountered in flight control that are nonaffine in the control input. The essence of the approach is to differentiate the function that is nonlinear in the control input and obtain an increased-order system that is linear in the derivative of the control signal and can be used as a new control variable. A systematic procedure is developed and related theoretical and practical issues are discussed. The proposed procedure, referred to as the controller for nonaffine plants, is developed for several cases of nonaffine models with unknown parameters. It is shown that the key aspect in the adaptive control design is the definition of the estimate of the derivative of system's state, which results in a convenient error model from which the adaptive laws can be written in a straightforward manner. The proposed approach is tested using a three-degree-of-freedom simulation of a typical fighter aircraft and is shown to result in a substantially improved system response.

I. Introduction

IN THE past decade there has been significant progress in the area of control design for nonlinear plants. Isidori[1] developed important results related to the geometric approach for analysis and control design of nonlinear plants. An overview of available nonlinear control techniques is given by Nijmeijer and Van Der Shaft.[2] Many of these results have been extended to the case of nonlinear plants with parametric uncertainty,[1–7] and systematic design procedures have also been developed.[8] The problem of adaptive control of nonlinear plants in which unknown parameters appear in a nonlinear fashion has also been addressed in the literature,[9–11] and several control design procedures have been developed. Most of the adaptive control methods developed in this context are applicable to nonlinear plant models that are linear in unknown parameters and affine in the control input vector u, that is, characterized by u appearing linearly in the state equation.

The problem of controlling the plants characterized by models that are nonaffine in the control input vector is a difficult one.[2,12–14] An approach widely used in practice is that based on linearization of the nonlinear plant model around an operating point. In aircraft control, the nonlinear model of aircraft dynamics is generally nonaffine in u and is commonly linearized around a trim point, that is, an operating point dependent on the current flight regime. Whereas the linearization may result in the design of sufficiently accurate controllers in the case of stabilization around the operating point, in the case of tracking of desired trajectories the problem becomes much more difficult, because the linearized model is time-varying. Using a fixed linear controller in such a case can result in an unacceptable response and even in instability of the closed-loop system. Hence, there is a clear need for the development of systematic control design techniques for nonlinear models that are nonaffine in u and that are suitable for the case of tracking of desired trajectories.

One nonlinear approach to this problem is that based on directly inverting the nonlinear function of u on a domain. Although the existence of an inverse function can be guaranteed by the implicit function theorem,[2] it is generally difficult to prescribe a technique to actually obtain such an inverse.

In another approach,[12] a nonlinear state equation nonaffine in u is transformed into an augmented state-space model in which the new control variable appears in a linear fashion. This is accomplished by differentiating the original state equation once so that the resulting augmented state equation is linear in $\dot{u}$, the time derivative of u, which is then used as the new control input. We believe that there is a lot of potential in this approach. However, systematic procedures have to be developed to assure stability of the overall closed-loop system and boundedness of all the signals and to address the case when the parameters of the nonlinear model are unknown. The conditions under which the transformation remains nonsingular also have to be established. In this paper we will present such a procedure.

The paper is organized as follows. The problem to be addressed is stated in Sec. II, and a solution for the first-order plant with known parameters is discussed. Section III contains the adaptive control design in the case of first-order plants with unknown parameters, and an extension to the case of higher-order plants. Application to a model of unmanned aerial vehicle (UAV) dynamics is discussed in Sec. IV, and conclusions are given in Sec. V, followed by the Appendix.

II. Motivation and Problem Statement

In the following sections, boldface lower-case letters denote vectors, capital letters denote matrices, and normal-weight lower-case letters denote scalars.

To motivate the proposed approach, in this section the focus is on a class of first-order plants described by the following scalar model:

$$\dot{x} = f(x, u) \tag{1}$$

where $x : \mathbb{R}^+ \to \mathbb{R}$ and $u : \mathbb{R}^+ \to \mathbb{R}$ denote, respectively, the state and input of the plant; $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable with respect to both x and u; and $\mathbb{R}^+$ denotes the set of positive real numbers.

Qualitatively speaking, the objective is to find $u(t)$ such that $x(t) - x^*(t)$ is small in some sense, where $x^*(t)$ denotes the desired plant dynamics. The main difficulty in achieving this is the fact that u appears in Eq. (1) in a nonlinear fashion. In some cases it is possible to find an inverse $\varphi(x)$ of $f(x, u)$ on a domain, such that $f(x, \varphi(x)) = v$, where v denotes some desired dynamics. To

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establish the existence of such an inverse (by using, for instance, the implicit function theorem), let $\mathcal{S}_{xu} = \{(x, u) : \|x\| \le c_x, \|u\| \le c_u\}$ denote a domain of interest in the (x, u) space, where c_x and c_u are known constants. The following assumption is now made:

Assumption 1: $|\partial f(x, u)/\partial u| \ge \alpha > 0, \forall (x, u) \in \mathcal{S}_{xu}$.

Even though the existence of the inverse is guaranteed under this assumption, in practice it is not always clear how to invert f, and some approximation is needed to arrive at a suitable control strategy.

In this paper the focus will be on an approach for solving the tracking control design problem for the plant [Eq. (1)] without any approximation. The approach is based on taking the time derivative of Eq. (1) so that $\dot{u}$ appears linearly in the resulting equation; that is,

$$\ddot{x} = \frac{\partial f(x, u)}{\partial x} f(x, u) + \frac{\partial f(x, u)}{\partial u} \dot{u} \tag{2}$$

Under Assumption 1, the control law is now chosen as

$$\dot{u} = \frac{1}{\partial f(x, u)/\partial u} \left(-\frac{\partial f(x, u)}{\partial x} f(x, u) + v \right) \tag{3}$$

resulting in $\ddot{x} = v$, where v can now be chosen to achieve the desired objective. For instance, if $x^*(t)$ is twice differentiable, v can be chosen as

$$v = \ddot{x}^* - k_1(x - x^*) - k_2(\dot{x} - \dot{x}^*) \tag{4}$$

where $k_1 > 0$ and $k_2 > 0$. Let $e = x - x^*$; then the closed-loop system is governed by $\ddot{e} + k_2\dot{e} + k_1 e = 0$, that is, the tracking objective is achieved exponentially. It is noted that in Eq. (4) $\dot{x}$ can be implemented using the right-hand side of Eq. (1).

The next objective is to show that the signals in the system, in particular $u(t)$, are bounded. This is discussed in the following proposition:

Proposition 1: All signals in the system (1), (3), and (4) are bounded.

Proof: Boundedness of e and $\dot{e}$ also implies that x and $\dot{x}$ are bounded (because x^* and $\dot{x}^*$ are bounded). Hence, the main issue here is to establish the boundedness of u, which is a nontrivial problem because $u(t)$ is adjusted dynamically using Eq. (3). It is first noted that Eq. (1) implies that, because $\dot{x}$ is bounded, $f(x, u)$ is also bounded. Next it needs to be shown that if x and $f(x, u)$ are bounded, so is u. Since f is continuously differentiable, arbitrary bounded values x_0 and u_0 are chosen, and by adding and subtracting terms $f(x_0, u_0)$ and $f(x, u_0)$, f is rewritten as

$$f(x, u) = f(x_0, u_0) + (f(x, u_0) - f(x_0, u_0)) + (f(x, u) - f(x, u_0))$$

Using the mean-value theorem,[15] it follows that

$$f(x, u_0) - f(x_0, u_0) = \frac{\partial f(\xi_x, u)}{\partial x}(x - x_0)$$

$$f(x, u) - f(x, u_0) = \frac{\partial f(x, \xi_u)}{\partial u}(u - u_0)$$

where $\xi_x \in [x_0, x]$ and $\xi_u \in [u_0, u]$. Hence,

$$f(x, u) = f(x_0, u_0) + \frac{\partial f(\xi_x, u_0)}{\partial x}(x - x_0) + \frac{\partial f(x, \xi_u)}{\partial u}(u - u_0)$$

It is also noted that, because x_0 and u_0 are bounded, so is $f(x_0, u_0)$. Because f is continuously differentiable with respect to x, $\partial f(\xi_x, u_0)/\partial x$ is also bounded. Hence, it follows that there exists a constant $c > 0$ such that

$$\left| \frac{\partial f(x, \xi_u)}{\partial u}(u - u_0) \right| \le c, \qquad \forall (x, u) \in \mathcal{S}_{xu}$$

Invoking Assumption 1, it follows that

$$|u - u_0| \le c/\alpha, \qquad \forall (x, u) \in \mathcal{S}_{xu}$$

Because $|u - u_0| \ge |u| - |u_0|$, and because u_0 is bounded, we can conclude that u is bounded as well. □

Comment: One of the important questions that arises regarding the preceding control strategy is that of assuring that $(x(t), u(t)) \in \mathcal{S}_{xu}$ for all time, which in turn assures that Assumption 1 is satisfied for all time. This is discussed as follows, based on the fact that the closed-loop system is linear and governed by the equation $\ddot{x} + k_2\dot{x} + k_1 x = \ddot{x}^* + k_2\dot{x}^* + k_1 x^*$.

1) Because $x^*(t)$, $\dot{x}^*(t)$, and $\ddot{x}^*(t)$ are known functions of time, for a known set of initial conditions $\{(x(0), \dot{x}(0)) : |x(0)| \le \alpha_1, |\dot{x}(0)| \le \alpha_2\}$, it is possible to calculate the bounds on $x(t)$ and $\dot{x}(t)$ such that $|x(t)| \le \bar{c}_x$ and $|\dot{x}(t)| \le \bar{c}_{dx}$ for all time. Here α_1, α_2, $\bar{c}_x$, and $\bar{c}_{dx}$ denote suitably chosen constants.

2) The latter condition on $\dot{x}(t)$ implies that $|f(x(t), u(t))| \le \bar{c}_{dx}$ for all time. Because $x(t)$ is bounded and its bound is known, the latter inequality can be solved analytically or numerically to find $\bar{c}_u$ such that $|u(t)| \le \bar{c}_u$ for all time.

3) Recalling the definition of the set $\mathcal{S}_{xu}$, it is seen that the condition of Assumption 1 will be satisfied if

$$\bar{c}_x < c_x, \qquad \bar{c}_u < c_u \tag{5}$$

4) Because the bounds on the states of the closed-loop system depend on initial conditions and $x^*(t)$ and its derivatives, it is possible to find a combination of these such that conditions (5) are satisfied. Hence, Assumption 1 limits the set of allowable initial conditions for x and $\dot{x}$ and the reference trajectories that can be followed.

It is seen that the aforementioned approach solves a difficult problem of control design for plants that are nonaffine in u. The idea of adding u to the state space and getting affinity in $\dot{u}$ is not new.[12] However, to the best of the authors' knowledge, this idea has not been exploited to develop a systematic design procedure for nonaffine plant models when there is uncertainty present in the description of the plant dynamics. In this paper, such a procedure will be developed and referred to as the controller for nonaffine plants (CNAP).

III. Adaptive Controller for Nonaffine Plants Design

In this section the focus will be on the control design for a class of nonaffine models containing constant unknown parameters that appear linearly in the model. Three cases will be considered: 1) a first-order single-input case; 2) a higher-order, relative degree one, multiple-input case; and 3) a higher-order, relative degree d, single-input case.

In all cases it is assumed that the plant contains a constant unknown parameter vector $\boldsymbol{p}$ that lies in a known set $\mathcal{S}_p$, and that the state vector of the system is accessible.

A. First-Order Plants

Let a first-order single-input system be described by

$$\dot{x} = \sum_{i=1}^{N} p_i f_i(x, u) \triangleq \boldsymbol{\omega}^T(x, u)\boldsymbol{p} \tag{6}$$

where p_i, $i = 1, 2, \ldots, N$ are constant unknown parameters; $f_i(x, u)$, $i = 1, 2, \ldots, N$ are continuously differentiable functions of their arguments; $\boldsymbol{p} \triangleq [p_1, p_2, \ldots, p_N]^T$; and $\boldsymbol{\omega}(x, u) \triangleq [f_1(x, u), f_2(x, u), \ldots, f_N(x, u)]^T$. Let the following functions also be defined:

$$\boldsymbol{\psi}_x(x, u) \triangleq \left[\frac{\partial f_1(x, u)}{\partial x}, \frac{\partial f_2(x, u)}{\partial x}, \ldots, \frac{\partial f_N(x, u)}{\partial x} \right]^T$$

$$\boldsymbol{\psi}_u(x, u) \triangleq \left[\frac{\partial f_1(x, u)}{\partial u}, \frac{\partial f_2(x, u)}{\partial u}, \ldots, \frac{\partial f_N(x, u)}{\partial u} \right]^T$$

It is seen that the relationship between $\boldsymbol{\psi}_x(x, u)$ and $\boldsymbol{\psi}_u(x, u)$ and the Jacobians $J_x(x, u, \boldsymbol{p})$ and $J_u(x, u, \boldsymbol{p})$ is as follows:

$$J_x(x, u, \boldsymbol{p}) \triangleq \frac{\partial \boldsymbol{\omega}^T(x, u)\boldsymbol{p}}{\partial x} = \boldsymbol{\psi}_x^T(x, u)\boldsymbol{p}$$

$$J_u(x, u, \boldsymbol{p}) \triangleq \frac{\partial \boldsymbol{\omega}^T(x, u)\boldsymbol{p}}{\partial u} = \boldsymbol{\psi}_u^T(x, u)\boldsymbol{p}$$

As discussed in the preceding section, the following assumption is needed to assure existence of a solution to the control problem.

Assumption 2: For any $\boldsymbol{p} \in \mathcal{S}_p$, $\psi_u^T(x, u)\boldsymbol{p} \geq \alpha > 0, \forall (x, u) \in \mathcal{S}_{xu}$.

In this section the design that solves the adaptive control problem for the plant, Eq. (6), is presented. The proposed approach is based on introducing the following variables:

$$\eta_1 = x \tag{7}$$

$$\eta_2 = \boldsymbol{\omega}^T(x, u)\hat{\boldsymbol{p}} \tag{8}$$

where $\hat{\boldsymbol{p}} \triangleq [\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_N]^T$ is an estimate of $\boldsymbol{p}$. Hence, η_2 is the estimate of $\dot{x}$. Let also

$$\phi \triangleq \hat{\boldsymbol{p}} - \boldsymbol{p}$$

Then, from Eqs. (6–8) it follows that

$$\begin{aligned} \dot{\eta}_1 &= \dot{x} \\ &= \boldsymbol{\omega}^T(\eta_1, u)\boldsymbol{p} \\ &= \boldsymbol{\omega}^T(\eta_1, u)\hat{\boldsymbol{p}} - \boldsymbol{\omega}^T(\eta_1, u)\phi \\ &= \eta_2 - \boldsymbol{\omega}^T(\eta_1, u)\phi \end{aligned} \tag{9}$$

Upon differentiating Eq. (8), one obtains

$$\begin{aligned} \dot{\eta}_2 &= \boldsymbol{\omega}^T(\eta_1, u)\dot{\hat{\boldsymbol{p}}} + \psi_{\eta_1}^T(\eta_1, u)\hat{\boldsymbol{p}}\dot{\eta}_1 + \psi_u^T(\eta_1, u)\hat{\boldsymbol{p}}\dot{u} \\ &= \boldsymbol{\omega}^T(\eta_1, u)\dot{\hat{\boldsymbol{p}}} + \psi_{\eta_1}^T(\eta_1, u)\hat{\boldsymbol{p}}\big(\eta_2 - \boldsymbol{\omega}^T(\eta_1, u)\phi\big) + \psi_u^T(\eta_1, u)\hat{\boldsymbol{p}}\dot{u} \end{aligned}$$

where $\psi_{\eta_1}(\eta_1, u) \triangleq \psi_x(x, u)$. It is noted that $\dot{u}$ has appeared linearly in the latter equation.

Let the reference model be of the form

$$\dot{\eta}_1^* = \eta_2^*, \qquad \dot{\eta}_2^* = -k_1\eta_1^* - k_2\eta_2^* + k_1 r \tag{10}$$

Now the control law is chosen as

$$\dot{u} = \frac{1}{\psi_u^T(\eta_1, u)\hat{\boldsymbol{p}}}\big(-\boldsymbol{\omega}^T(\eta_1, u)\dot{\hat{\boldsymbol{p}}} - \psi_{\eta_1}^T(\eta_1, u)\hat{\boldsymbol{p}}\eta_2 - k_1\eta_1 - k_2\eta_2 + k_1 r\big) \tag{11}$$

which yields

$$\dot{\eta}_2 = -k_1\eta_1 - k_2\eta_2 + k_1 r - \psi_{\eta_1}^T(\eta_1, u)\hat{\boldsymbol{p}}\boldsymbol{\omega}^T(\eta_1, u)\phi \tag{12}$$

Equations (9), (12), and (10) yield an error model of the form

$$\dot{\boldsymbol{e}}_\eta = \Lambda\boldsymbol{e}_\eta - \Theta(\eta_1, u, \hat{\boldsymbol{p}})\phi \tag{13}$$

where

$$\boldsymbol{e}_\eta \triangleq \begin{bmatrix}\eta_1 - \eta_1^*, & \eta_2 - \eta_2^*\end{bmatrix}^T, \qquad \Lambda \triangleq \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}$$

$$\Theta(\eta_1, u, \hat{\boldsymbol{p}}) \triangleq \begin{bmatrix} \psi^T(\eta_1, u) \\ \psi_{\eta_1}^T(\eta_1, u)\hat{\boldsymbol{p}}\boldsymbol{\omega}^T(\eta_1, u) \end{bmatrix}$$

Because Λ is asymptotically stable, for a given matrix $Q = Q^T > 0$ the solution $P = P^T > 0$ of the Lyapunov matrix equation $\Lambda^T P + P\Lambda = -Q$ can be readily found. The following theorem can now be established.

Theorem 1: Adaptive laws

$$\dot{\hat{\boldsymbol{p}}} = \dot{\phi} = \Gamma\Theta^T(\eta_1, u, \hat{\boldsymbol{p}})P\boldsymbol{e}_\eta \tag{14}$$

where $\Gamma = \mathrm{diag}[\gamma_1, \gamma_2, \gamma_3]$, $\gamma_i > 0 (i = 1, 2, 3)$, and where $\hat{\boldsymbol{p}}$ is adjusted using adaptive algorithms with projection (see the Appendix) to assure that $\hat{\boldsymbol{p}} \in \mathcal{S}_p$ for all time, result in the boundedness of all signals in the system (13), (14) and, in addition, $\lim_{t\to\infty} \boldsymbol{e}_\eta(t) = 0$.

Proof: It is first noted that, because the unknown parameters are constant, $\dot{\phi} = \dot{\hat{\boldsymbol{p}}}$. A tentative Lyapunov function is chosen as

$$V(\boldsymbol{e}_\eta, \phi) = \tfrac{1}{2}\big(\boldsymbol{e}_\eta^T P\boldsymbol{e}_\eta + \phi^T\Gamma^{-1}\phi\big)$$

Upon taking its first derivative along the motions of Eqs. (13) and (14) one obtains

$$\begin{aligned} \dot{V}(\boldsymbol{e}_\eta, \phi) &= \tfrac{1}{2}\boldsymbol{e}_\eta^T(\Lambda^T P + P\Lambda)\boldsymbol{e}_\eta - \boldsymbol{e}_\eta^T P\Theta(\eta_1, u, \hat{\boldsymbol{p}})\phi + \phi^T\Gamma^{-1}\dot{\phi} \\ &\leq -\tfrac{1}{2}\boldsymbol{e}_\eta^T Q\boldsymbol{e}_\eta \\ &\leq -\tfrac{1}{2}\lambda_Q\|\boldsymbol{e}_\eta\|^2 \leq 0, \qquad \forall(\boldsymbol{e}_\eta, \phi) \neq (0, 0) \end{aligned}$$

where λ_Q denotes the minimum eigenvalue of Q, and $\boldsymbol{e}_\eta^T Q\boldsymbol{e}_\eta \geq \lambda_Q\|\boldsymbol{e}_\eta\|^2$. It is noted that, as shown in the Appendix, in the case of adaptive algorithms with projection it follows that $\phi^T\Gamma^{-1}\dot{\phi} \leq \phi^T\Theta^T(\eta_1, u, \hat{\boldsymbol{p}})P\boldsymbol{e}_\eta$.

Because $\dot{V} \leq 0$, it follows that $\boldsymbol{e}_\eta$ and ϕ are bounded. Upon integrating $\dot{V}$ one obtains

$$V(0) - V(\infty) \geq \lambda_Q\int_0^\infty \|\boldsymbol{e}_\eta(t)\|^2\,\mathrm{d}t$$

Because $V(0)$ and $V(\infty)$ are bounded, it follows that $\boldsymbol{e}_\eta \in \mathcal{L}^2$, where $\mathcal{L}^2$ denotes the space of square-integrable functions. The next objective is to show that $\dot{\boldsymbol{e}}_\eta$ is also bounded.

Because $\boldsymbol{e}_\eta$ and $\boldsymbol{\eta}^* = [\eta_1^*, \eta_2^*]^T$ are bounded, it follows that η_1 and η_2 are also bounded. Hence, from Eqs. (7) and (8) it follows that x and $\boldsymbol{\omega}^T(x, u)\hat{\boldsymbol{p}}$ are bounded. By Assumption 2 and an argument similar to that in the proof of Proposition 1, it follows that u is bounded. It can now be concluded that $\dot{\boldsymbol{e}}_\eta$ is bounded. From Barbalat's lemma (see, e.g., Ref. 16), it follows that $\lim_{t\to\infty} \boldsymbol{e}_\eta(t) = 0$, which completes the proof. □

Comment: It is noted that the control law (11) contains the term $\dot{\hat{\boldsymbol{p}}}$. To implement this control law, $\dot{\hat{\boldsymbol{p}}}$ is replaced by the adaptive law (14). Hence,

$$\begin{aligned} \dot{u} = \frac{1}{\psi_u^T(\eta_1, u)\hat{\boldsymbol{p}}}\big(&-\boldsymbol{\omega}^T(\eta_1, u)\Gamma\Theta^T(\eta_1, u, \hat{\boldsymbol{p}})P\boldsymbol{e}_\eta \\ &-\psi_{\eta_1}^T(\eta_1, u)\hat{\boldsymbol{p}}\eta_2 - k_1\eta_1 - k_2\eta_2 + k_1 r\big) \end{aligned}$$

which can be readily implemented.

It can be concluded that the proposed method solves the tracking control problem for the plant [Eq. (6)] in the presence of unknown parameters. The key aspect in the adaptive control design is the definition of η_2, the estimate of the derivative of system's state $\dot{x}$, which results in a convenient error model from which the adaptive laws can be written in a straightforward manner.

An extension of the adaptive control design to higher-order plants is described in the following sections.

B. Higher-Order Plants

To generalize the results for first-order plants, two cases are considered next: 1) relative degree one, multiple-input plants, and 2) relative degree d, single-input plants.

1. *Relative Degree One, Multiple Inputs*

The system is described by

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{N} p_i\boldsymbol{f}_i(\boldsymbol{x}, \boldsymbol{u}) \triangleq \Omega(\boldsymbol{x}, \boldsymbol{u})\boldsymbol{p} \tag{15}$$

where $\boldsymbol{x}$ and $\boldsymbol{u}$ have the same dimension m; p_i, $i = 1, 2, \ldots, N$ are constant unknown parameters; $\boldsymbol{f}_i(\boldsymbol{x}, \boldsymbol{u})$, $i = 1, 2, \ldots, N$ are continuously differentiable functions of their arguments; $\boldsymbol{p} \triangleq [p_1, p_2, \ldots, p_N]^T$; and $\Omega(x, u) \triangleq [\boldsymbol{f}_1(x, u), \boldsymbol{f}_2(x, u), \ldots, \boldsymbol{f}_N(x, u)]$. Also let the Jacobian matrices be defined as

$$J_x(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) \triangleq \frac{\partial\Omega(\boldsymbol{x}, \boldsymbol{u})\boldsymbol{p}}{\partial\boldsymbol{x}} = \sum_{i=1}^{N} p_i\frac{\partial\boldsymbol{f}_i(\boldsymbol{x}, \boldsymbol{u})}{\partial\boldsymbol{x}}$$

$$J_u(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) \triangleq \frac{\partial\Omega(\boldsymbol{x}, \boldsymbol{u})\boldsymbol{p}}{\partial\boldsymbol{u}} = \sum_{i=1}^{N} p_i\frac{\partial\boldsymbol{f}_i(\boldsymbol{x}, \boldsymbol{u})}{\partial\boldsymbol{u}}$$

Existence of a control solution is assured by the following assumption:

Assumption 3: For any $\boldsymbol{p} \in \mathcal{S}_p$, the Jacobian matrix $J_u(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p})$ satisfies the condition

$$\det[J_u(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p})] \geq \alpha > 0, \qquad \forall (x, u) \in \mathcal{S}_{xu}$$

The following variables are introduced next:

$$\boldsymbol{\eta}_1 = \boldsymbol{x} \tag{16}$$

$$\boldsymbol{\eta}_2 = \Omega(\boldsymbol{x}, \boldsymbol{u})\hat{\boldsymbol{p}} \tag{17}$$

$$\boldsymbol{\phi} = \hat{\boldsymbol{p}} - \boldsymbol{p} \tag{18}$$

It follows that

$$\dot{\boldsymbol{\eta}}_1 = \boldsymbol{\eta}_2 - \Omega(\boldsymbol{\eta}_1, \boldsymbol{u})\boldsymbol{\phi} \tag{19}$$

$$\dot{\boldsymbol{\eta}}_2 = \Omega(\boldsymbol{\eta}_1, \boldsymbol{u})\dot{\hat{\boldsymbol{p}}} + J_{\eta_1}(\boldsymbol{\eta}_1, \boldsymbol{u}, \hat{\boldsymbol{p}})[\boldsymbol{\eta}_2 - \Omega(\boldsymbol{\eta}_1, \boldsymbol{u})\boldsymbol{\phi}] + J_u(\boldsymbol{\eta}_1, \boldsymbol{u}, \hat{\boldsymbol{p}})\dot{\boldsymbol{u}} \tag{20}$$

where $J_{\eta_1}(\boldsymbol{\eta}_1, \boldsymbol{u}, \hat{\boldsymbol{p}}) \triangleq J_x(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p})$. Let the reference model be of the form

$$\dot{\boldsymbol{\eta}}_1^* = \boldsymbol{\eta}_2^*, \qquad \dot{\boldsymbol{\eta}}_2^* = -K_1\boldsymbol{\eta}_1^* - K_2\boldsymbol{\eta}_2^* + K_1\boldsymbol{r} \tag{21}$$

where K_1 and K_2 are chosen such that the following matrix is asymptotically stable:

$$\Lambda \triangleq \begin{bmatrix} 0 & I \\ -K_1 & -K_2 \end{bmatrix}$$

The control law is chosen as

$$\dot{\boldsymbol{u}} = J_u^{-1}(\boldsymbol{\eta}_1, \boldsymbol{u}, \hat{\boldsymbol{p}})\big(-\Omega(\boldsymbol{\eta}_1, \boldsymbol{u})\dot{\hat{\boldsymbol{p}}} - J_{\eta_1}(\boldsymbol{\eta}_1, \boldsymbol{u}, \hat{\boldsymbol{p}})\boldsymbol{\eta}_2 - K_1\boldsymbol{\eta}_1 - K_2\boldsymbol{\eta}_2 + K_1\boldsymbol{r}\big) \tag{22}$$

which leads to the error equation

$$\dot{\boldsymbol{e}}_\eta = \Lambda\boldsymbol{e}_\eta - \Theta(\boldsymbol{\eta}_1, \boldsymbol{u}, \hat{\boldsymbol{p}})\boldsymbol{\phi} \tag{23}$$

where

$$\boldsymbol{e}_\eta \triangleq \begin{bmatrix}\boldsymbol{\eta}_1^T, & \boldsymbol{\eta}_2^T\end{bmatrix}^T$$

$$\Theta(\boldsymbol{\eta}_1, \boldsymbol{u}, \hat{\boldsymbol{p}}) \triangleq \begin{bmatrix} \Omega(\boldsymbol{\eta}_1, \boldsymbol{u}) \\ J_{\eta_1}(\boldsymbol{\eta}_1, \boldsymbol{u}, \hat{\boldsymbol{p}})\Omega(\boldsymbol{\eta}_1, \boldsymbol{u}) \end{bmatrix}$$

The adaptive law (with projection to maintain that $\hat{\boldsymbol{p}} \in \mathcal{S}_p$)

$$\dot{\hat{\boldsymbol{p}}} = \dot{\boldsymbol{\phi}} = \Gamma\Theta^T(\boldsymbol{\eta}_1, \boldsymbol{u}, \hat{\boldsymbol{p}})P\boldsymbol{e}_\eta \tag{24}$$

(where, as before, P is the solution to the Lyapunov equation $\Lambda^T P + P\Lambda = -Q$ for a given $Q = Q^T > 0$) will achieve the tracking objective, and the proof is omitted.

Comment: The proposed technique can be readily extended to the overactuated case, that is, the case when the number of control inputs exceeds the number of controlled variables. Due to the importance of this case in the flight control design, a related extension of the proposed approach is presented next.

Let the plant dynamics be described by the equation

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \tag{25}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^m$, and $m > n$. In this case the following assumption is made:

Assumption 4: $\det[J_u(\boldsymbol{x}, \boldsymbol{u})J_u^T(\boldsymbol{x}, \boldsymbol{u})] \geq \alpha > 0, \quad \forall (x, u) \in \mathcal{S}_{xu}$.

Applying the same approach as for the baseline CNAP, one obtains

$$\ddot{\boldsymbol{x}} = J_x(\boldsymbol{x}, \boldsymbol{u})\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) + J_u(\boldsymbol{x}, \boldsymbol{u})\dot{\boldsymbol{u}} \tag{26}$$

The only difference with respect to the case of the fully actuated system is that the size of the Jacobian J_u now is $n \times m$. Hence, the control law can be chosen as

$$\dot{\boldsymbol{u}} = WJ_u^T\big(J_uWJ_u^T\big)^{-1}(-J_x\boldsymbol{f} - K_1\boldsymbol{x} - K_2\dot{\boldsymbol{x}} + K_1\boldsymbol{r}) \tag{27}$$

where $W = W^T > 0$ denotes the control allocation matrix.

The adaptive controller in the case of unknown parameters can be designed along exactly the same lines as in the case when $m = n$.

2. Relative Degree d*, Single Input*

In this case the system is described by

$$\dot{x}_1 = x_2 \tag{28}$$

$$\dot{x}_2 = x_3 \tag{29}$$

$$\vdots$$

$$\dot{x}_d = \sum_{i=1}^{N} p_i f_i(\boldsymbol{x}, u) \triangleq \boldsymbol{\omega}^T(\boldsymbol{x}, u)\boldsymbol{p} \tag{30}$$

$$\dot{\boldsymbol{x}}_Z = \boldsymbol{h}(\boldsymbol{x}, u) \tag{31}$$

$$y = x_1 \tag{32}$$

where $\boldsymbol{x} = [x_1, x_2, \ldots, x_d]^T$, $\boldsymbol{x}_Z = [x_{d+1}, \ldots, x_n]^T$, p_i, $i = 1, 2, \ldots, N$ are constant unknown parameters; $f_i(\boldsymbol{x}, u)$, $i = 1, 2, \ldots, N$ and $h(\boldsymbol{x}, u)$ are continuously differentiable functions of their arguments; $\boldsymbol{p} \triangleq [p_1, p_2, \ldots, p_N]^T$; and $\boldsymbol{\omega}(\boldsymbol{x}, u) \triangleq [f_1(\boldsymbol{x}, u), f_2(\boldsymbol{x}, u), \ldots, f_N(\boldsymbol{x}, u)]^T$. Define also

$$\Psi(\boldsymbol{x}, u) \triangleq \begin{bmatrix} \dfrac{\partial f_1(\boldsymbol{x}, u)}{\partial \boldsymbol{x}} \\ \dfrac{\partial f_2(\boldsymbol{x}, u)}{\partial x} \\ \vdots \\ \dfrac{\partial f_N(\boldsymbol{x}, u)}{\partial \boldsymbol{x}} \end{bmatrix}$$

$$\boldsymbol{\omega}_u(\boldsymbol{x}, u) \triangleq \left[\frac{\partial f_1(\boldsymbol{x}, u)}{\partial u}, \frac{\partial f_2(\boldsymbol{x}, u)}{\partial u}, \ldots, \frac{\partial f_N(\boldsymbol{x}, u)}{\partial u}\right]^T$$

The objective is to design a control law $u(t)$ such that the error $y(t) - y^*(t) = x_1(t) - x_1^*(t)$ tends to zero asymptotically despite parametric uncertainty. For this the following assumption is needed:

Assumption 5: The system (28–32) satisfies the following. 1) For any $\boldsymbol{p} \in \mathcal{S}_p$, $\boldsymbol{\omega}_u^T(x, u)\boldsymbol{p} \geq \alpha > 0$, $\forall (x, u) \in \mathcal{S}_{xu}$; and 2) let $\bar{\boldsymbol{x}} \triangleq [x_1, \ldots, x_d]^T$. The zero dynamics $\boldsymbol{x}_Z(t)$, governed by

$$\dot{\boldsymbol{x}}_Z = h\left(\left[\bar{\boldsymbol{x}}^T, \boldsymbol{x}_Z^T\right]^T, u\right)$$

are bounded for bounded $\bar{\boldsymbol{x}}(t)$ and $u(t)$.

It is noted that condition 2) is a stronger version of the "minimum-phase" requirement for nonlinear systems.

To solve the adaptive control problem, the following variables are introduced:

$$\eta_1 = x_1$$
$$\eta_2 = x_2$$
$$\vdots$$
$$\eta_d = x_d$$
$$\eta_{d+1} = \boldsymbol{\omega}^T(\boldsymbol{x}, u)\hat{\boldsymbol{p}}$$
$$\eta_{d+2} = x_{d+1}$$
$$\vdots$$
$$\eta_{n+1} = x_n$$

Define also

$$\boldsymbol{\eta} \triangleq [\eta_1, \eta_2, \ldots, \eta_{n+1}]^T$$

$$\bar{\boldsymbol{\eta}} \triangleq [\eta_1, \eta_2, \ldots, \eta_d, \eta_{d+2}, \ldots, \eta_{n+1}]^T = \boldsymbol{x}$$

$$\boldsymbol{\eta}_Z \triangleq [\eta_{d+2}, \ldots, \eta_{n+1}]^T = \boldsymbol{x}_Z$$

$$\boldsymbol{\phi} \triangleq \hat{\boldsymbol{p}} - \boldsymbol{p}$$

It follows that

$$\begin{aligned}
\dot{\eta}_1 &= \eta_2 \\
\dot{\eta}_2 &= \eta_3 \\
&\vdots \\
\dot{\eta}_d &= \eta_{d+1} - \boldsymbol{\omega}^T(\bar{\boldsymbol{\eta}}, u)\boldsymbol{\phi} \\
\dot{\eta}_{d+1} &= \boldsymbol{\omega}^T(\bar{\boldsymbol{\eta}}, u)\dot{\hat{\boldsymbol{p}}} + \hat{\boldsymbol{p}}^T \Psi(\bar{\boldsymbol{\eta}}, u)\Big[\tilde{\boldsymbol{\eta}}^T, \big[\eta_{d+1} - \boldsymbol{\omega}^T(\bar{\boldsymbol{\eta}}, u)\boldsymbol{\phi}\big], \\
&\quad \boldsymbol{h}^T(\bar{\boldsymbol{\eta}}, u)\Big]^T + \boldsymbol{\omega}_u^T(\bar{\boldsymbol{\eta}}, u)\hat{\boldsymbol{p}}\dot{u} \\
\dot{\boldsymbol{\eta}}_Z &= \boldsymbol{h}(\bar{\boldsymbol{\eta}}, u)
\end{aligned}$$

where $\tilde{\boldsymbol{\eta}} \triangleq [\eta_2, \eta_3, \ldots, \eta_d]^T$. Since $\bar{\boldsymbol{\eta}}$ is a subset of $\boldsymbol{\eta}$, in the following the former will be replaced by the latter when used as an argument of a function.

Let the reference model be of the form

$$\begin{aligned}
\dot{\eta}_1^* &= \eta_2^* \\
\dot{\eta}_2^* &= \eta_3^* \\
&\vdots \\
\dot{\eta}_{d+1}^* &= -\sum_{i=1}^{d+1} k_i \eta_i^* + k_1 r
\end{aligned}$$

The control law is chosen as

$$\dot{u} = \frac{1}{\boldsymbol{\omega}_u^T(\boldsymbol{\eta}, u)\hat{\boldsymbol{p}}}\Bigg(-\boldsymbol{\omega}^T(\boldsymbol{\eta}, u)\dot{\hat{\boldsymbol{p}}} - \hat{\boldsymbol{p}}^T\Psi(\boldsymbol{\eta}, u)\big[\tilde{\boldsymbol{\eta}}^T, \eta_{d+1}, \boldsymbol{h}^T(\boldsymbol{\eta}, u)\big]^T - \sum_{i=1}^{d+1} k_i \eta_i + k_1 r\Bigg)$$

which yields an error model of the form

$$\dot{\boldsymbol{e}}_\eta = \Lambda \boldsymbol{e}_\eta - \Theta(\boldsymbol{\eta}, u, \hat{\boldsymbol{p}})\boldsymbol{\phi}$$

where

$$\boldsymbol{e}_\eta \triangleq \big[\eta_1 - \eta_1^*, \eta_2 - \eta_2^*, \ldots, \eta_{d+1} - \eta_{d+1}^*\big]^T$$

$$\Lambda = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ & \vdots & & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -k_1 & -k_2 & -k_3 & \ldots & -k_{d+1} \end{bmatrix}$$

$$\Theta(\boldsymbol{\eta}, u, \hat{\boldsymbol{p}}) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \boldsymbol{\omega}^T(\boldsymbol{\eta}, u) \\ \hat{\boldsymbol{p}}^T \Psi_{(d)}(\boldsymbol{\eta}, u)\boldsymbol{\omega}^T \end{bmatrix}$$

in which $\Psi_{(d)}(\boldsymbol{\eta}, u)$ denotes the dth column of the matrix $\Psi(\boldsymbol{\eta}, u)$.

The adaptive law (with projection that maintains $\hat{\boldsymbol{p}} \in \mathcal{S}_p$)

$$\dot{\hat{\boldsymbol{p}}} = \dot{\boldsymbol{\phi}} = \Gamma \Theta^T(\boldsymbol{\eta}, u, \hat{\boldsymbol{p}}) P \boldsymbol{e}_\eta$$

will result in

$$\lim_{t\to\infty} |\eta_i(t) - \eta_i^*(t)| = 0, \qquad i = 1, 2, \ldots, d+1$$

Boundedness of $\boldsymbol{x}_Z$ follows from the Assumption 5, condition 2, on zero dynamics.

It is seen that the result for the first-order plants generalizes to the case of relative degree 1 multi-input plants in a relatively straightforward manner. The extension to the case of relative degree d multi-input/multi-output plants is involved due to a complex notation related to the state estimates.

IV. Simulations

In this section the objective is to evaluate the performance of the adaptive CNAP. The evaluation is carried out on the three-degree-of-freedom (3-DOF) model of UAV dynamics that can be found in Ref. 17.

The differential equations governing the point-mass UAV dynamics are given by

$$\dot{V} = g\left(\frac{T - D}{W} - \sin\gamma\right) \tag{33}$$

$$\dot{\gamma} = \frac{g}{V}(n\cos\mu - \cos\gamma) \tag{34}$$

$$\dot{\chi} = \frac{gn\sin\mu}{V\cos\gamma} \tag{35}$$

The three state variables are airspeed V, flight-path angle γ, and flight-path heading χ. The control inputs are thrust T, load factor n, and bank angle μ. In the preceding equations, g denotes the gravity acceleration constant, W is the weight, and D denotes the drag force, which can be approximated by a simple drag polar model of the form

$$D = 0.5\rho V^2 S C_{D_0} + \frac{2kn^2W^2}{\rho V^2 S} \tag{36}$$

The coefficients from Eq. (36) are defined in Table 1.

With respect to the preceding model, it is assumed that the desired load factor n and the applied load factor n_A are related as $n_A = k_n n$, where k_n denotes the load factor effectiveness coefficient such that $0 < \epsilon \le k_n \le 1$. In the nominal case $k_n = 1$, whereas the values less than one can result from actuator failures or battle damage. It is also assumed that there are three uncertain parameters: the parasite drag coefficient C_{D0}, induced-drag coefficient k, and k_n.

Let $\boldsymbol{x} = [V, \gamma, \chi]^T$, $\boldsymbol{u} = [T, n, \mu]^T$, and $\boldsymbol{p} = [p_1, p_2, p_3]^T$, where $p_1 = C_{D0}$, $p_2 = kk_n^2$, and $p_3 = k_n$. It is noted that the parameter p_2 is a product of parameters k and k_n. Then the 3-DOF dynamics model becomes

$$\dot{x}_1 = p_1 c_{11} x_1^2 + \frac{p_2 c_{12} u_2^2}{x_1^2} + c_{13}\sin(x_2) + c_{14} u_1 \tag{37}$$

$$\dot{x}_2 = \frac{p_3 c_{21} u_2 \cos u_3}{x_1} + \frac{c_{22}\cos x_2}{x_1} \tag{38}$$

$$\dot{x}_3 = \frac{p_3 c_{31} u_2 \sin u_3}{x_1 \cos x_2} \tag{39}$$

Table 1 UAV model parameters

Description	Value
Density, ρ	1.2251 km/m^3
Weight, W	14,515 kg
Reference area, S	37.16 m^2
Maximum thrust, $T_{\max}$	113,868.8 N
Maximum lift coefficient, $C_{L_{\max}}$	2.0
Maximum load factor, $n_{\max}$	7
Induced drag coefficient, k	0.1
Parasite drag coefficient, C_{D_0}	0.02

where $c_{11} = -0.5\rho S$, $c_{12} = 2W^2/(\rho S)$, $c_{13} = -g$, $c_{14} = g/W$, $c_{21} = g$, $c_{22} = -g$, and $c_{31} = g$ are known coefficients. Hence,

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{3} p_i \boldsymbol{f}_i(\boldsymbol{x}, \boldsymbol{u}) + \psi(\boldsymbol{x}, \boldsymbol{u}) = \Omega(\boldsymbol{x}, \boldsymbol{u})\boldsymbol{p} + \boldsymbol{f}_o(\boldsymbol{x}, \boldsymbol{u}) \tag{40}$$

where $\Omega(\boldsymbol{x}, \boldsymbol{u}) = [\boldsymbol{f}_1(\boldsymbol{x}, \boldsymbol{u}), \boldsymbol{f}_2(\boldsymbol{x}, \boldsymbol{u}), \boldsymbol{f}_3(\boldsymbol{x}, \boldsymbol{u})]$,

$$\boldsymbol{f}_1(\boldsymbol{x}, \boldsymbol{u}) = \begin{bmatrix} c_{11}x_1^2 \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{f}_2(\boldsymbol{x}, \boldsymbol{u}) = \begin{bmatrix} c_{12}u_2^2 / x_1^2 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{f}_3(\boldsymbol{x}, \boldsymbol{u}) = \begin{bmatrix} 0 \\ c_{21}u_2 \cos u_3 / x_1 \\ c_{31}u_2 \sin u_3/(x_1 \cos x_2) \end{bmatrix}$$

and where $\boldsymbol{f}_o(\boldsymbol{x}, \boldsymbol{u})$ is a known vector function:

$$\boldsymbol{f}_o(\boldsymbol{x}, \boldsymbol{u}) = \begin{bmatrix} c_{13} \sin x_2 + c_{14}u_1 \\ c_{22} \cos x_2 / x_1 \\ 0 \end{bmatrix}$$

A. Baseline Controller for Nonaffine Plants

The baseline CNAP is designed under an assumption that the parameter vector $\boldsymbol{p}$ is known. In this case, taking a derivative of Eq. (40) yields

$$\ddot{\boldsymbol{x}} = J_x(\boldsymbol{x}, \boldsymbol{u})(\Omega(\boldsymbol{x}, \boldsymbol{u})\boldsymbol{p} + \boldsymbol{f}_o(\boldsymbol{x}, \boldsymbol{u})) + J_u(\boldsymbol{x}, \boldsymbol{u})\dot{\boldsymbol{u}} \tag{41}$$

where

$$J_x(\boldsymbol{x}, \boldsymbol{u}) \triangleq \frac{\partial(\Omega(\boldsymbol{x}, \boldsymbol{u})\boldsymbol{p} + \boldsymbol{f}_o(\boldsymbol{x}, \boldsymbol{u}))}{\partial \boldsymbol{x}} = \begin{bmatrix} 2p_1c_{11}x_1 - 2p_2c_{12}u_2^2 / x_1^3 & c_{13}\cos x_2 & 0 \\ -(p_3c_{21}u_2\cos u_3 + c_{22}\cos x_2) / x_1^2 & -c_{22}\sin x_2/x_1 & 0 \\ -p_3c_{31}u_2 \sin u_3 / (x_1^2 \cos x_2) & c_{31}u_2 \sin u_3 \sin x_2 / (x_1 \cos^2 x_2) & 0 \end{bmatrix}$$

$$J_u(\boldsymbol{x}, \boldsymbol{u}) \triangleq \frac{\partial(\Omega(\boldsymbol{x}, \boldsymbol{u})\boldsymbol{p} + \boldsymbol{f}_o(\boldsymbol{x}, \boldsymbol{u}))}{\partial \boldsymbol{u}} = \begin{bmatrix} c_{14} & 2p_2c_{12}u_2 / x_1^2 & 0 \\ 0 & c_{21}\cos u_3/x_1 & -c_{21}u_2 \sin u_3/x_1 \\ 0 & c_{31}\sin u_3/(x_1\cos x_2) & c_{31}u_2\cos u_3/(x_1 \cos x_2) \end{bmatrix}$$

It is seen that $J_u(\boldsymbol{x}, \boldsymbol{u})$ is singular if $n = 0$, or $V = 0$, or $\gamma = \pm\pi/2$. However, these singular points are way beyond the normal operating regime of the aircraft.

The control law is now chosen as

$$\dot{\boldsymbol{u}} = J_u(\boldsymbol{x}, \boldsymbol{u})^{-1}(-J_x(\boldsymbol{x}, \boldsymbol{u})[\Omega(\boldsymbol{x}, \boldsymbol{u})\boldsymbol{p} + \boldsymbol{f}_o(\boldsymbol{x}, \boldsymbol{u})] - K_1\boldsymbol{x} - K_2\dot{\boldsymbol{x}} + K_1\boldsymbol{r}) \tag{42}$$

where K_1 and K_2 are diagonal matrices with positive elements, and $\boldsymbol{r}$ denotes a reference input. This control law yields $\ddot{\boldsymbol{x}} + K_2\dot{\boldsymbol{x}} + K_1\boldsymbol{x} = K_1\boldsymbol{r}$.

This control law is simulated next to evaluate its performance. In all simulations it is assumed that the objective is to assure that the forward velocity $V(t) = x_1(t)$ and flight-path angle $\gamma(t) = x_2(t)$ are regulated around their desired values $[300 \quad 0]^T$, while the heading angle $\chi(t) = x_3(t)$ follows an output of a reference model,

$$\dot{x}_{m1} = x_{m2} \tag{43}$$

$$\dot{x}_{m2} = -x_{m1} - 1.4x_{m2} + r_\chi \tag{44}$$

where $r_\chi(t)$ is a 30-deg heading angle doublet:

$$r_\chi(t) = \begin{cases} 0, & t <= 5 \\ 6(t-5), & 5 < t \le 10 \\ 30, & 10 < t \le 20 \\ 30(5 - 0.2t), & 20 < t \le 30 \\ -30, & 30 < t \le 40 \\ -30(9 - 0.2t), & 40 < t \le 45 \\ 0, & t > 45 \end{cases} \tag{45}$$

Initial conditions are chosen as follows: $V(0) = 300$ m/s, $\gamma(0) = 0$, and $\chi(0) = 0$. It is noted that the nominal values of the parameters are $p_1 = C_{D0} = 0.02$, $p_2 = k_o = kk_n^2 = 0.2$, and $p_3 = k_n = 1$.

Simulation 1: Baseline CNAP with Known Parameters

The response of the system with the baseline CNAP is shown in Fig. 1. The errors in $V(t)$, $\gamma(t)$, and $\chi(t)$ are shown in the upper portion of the figure, rather than the variables themselves. It is seen that the response is acceptable and that the control objective is achieved with all control inputs within the saturation bounds.

Simulation 2: Baseline CNAP with Unknown Parameters

The next simulation is focused on the case when the system is controlled by the baseline controller, but the parameters change in an unknown fashion. It is assumed that the parameters change at $t = 20$ s in the following manner:

$$C_{D0}(t) = C_{D0}^{\text{nom}}\left[2 - e^{-(t-20)}\right], \qquad k(t) = 200k^{\text{nom}}\left[1 - e^{-(t-20)}\right]$$
$$k_n(t) = 0.5\left[1 - e^{-(t-20)}\right] \tag{46}$$

The response of the system in this case is shown in Fig. 2. The superscript m denotes the outputs of the reference model. It is seen that the parameter variations lead to large errors.

B. Adaptive Controller for Nonaffine Plants

In the case of adaptive control, based on the discussion in the previous sections, the control law is chosen as

$$\boldsymbol{u} = J_u(\boldsymbol{x}, \boldsymbol{u})^{-1}(-\Omega(\boldsymbol{x}, \boldsymbol{u})\dot{\hat{\boldsymbol{p}}} - J_x(\boldsymbol{x}, \boldsymbol{u})[\Omega(\boldsymbol{x}, \boldsymbol{u})\hat{\boldsymbol{p}} + \boldsymbol{f}_o(\boldsymbol{x}, \boldsymbol{u})] - K_1\boldsymbol{\eta}_1 - K_2\boldsymbol{\eta}_2 + K_1\boldsymbol{r}) \tag{47}$$

where $\boldsymbol{\eta}_1 = \boldsymbol{x}$ and $\boldsymbol{\eta}_2 = \Omega(\boldsymbol{\eta}_1, \boldsymbol{u})\hat{\boldsymbol{p}} + \boldsymbol{f}_o(\boldsymbol{x}, \boldsymbol{u})$. The estimate $\hat{\boldsymbol{p}}$ is adjusted using

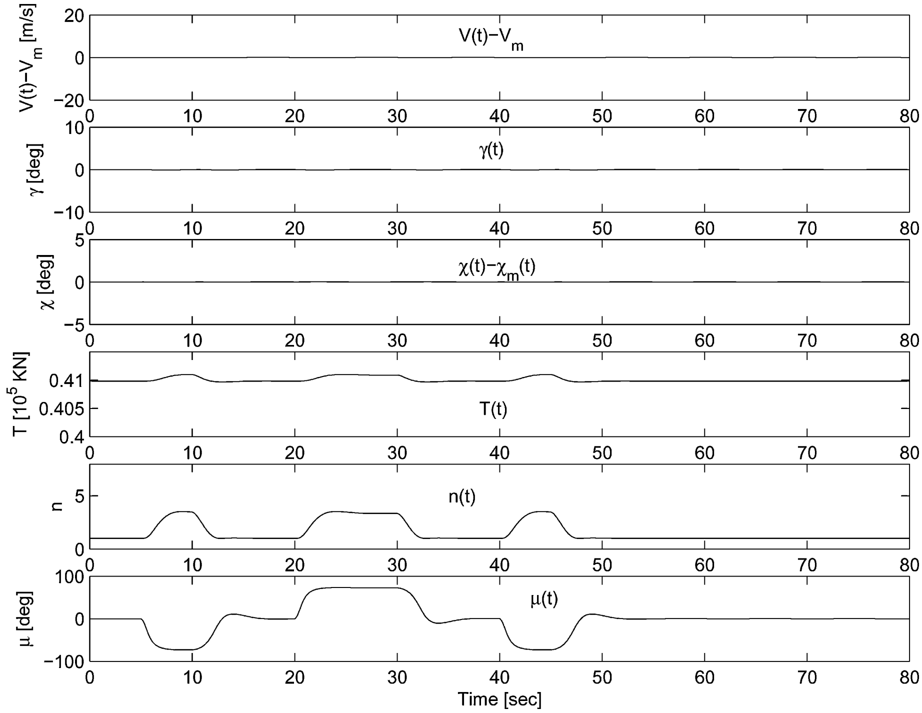

Fig. 1 System response with the baseline CNAP in the case of known *p*.

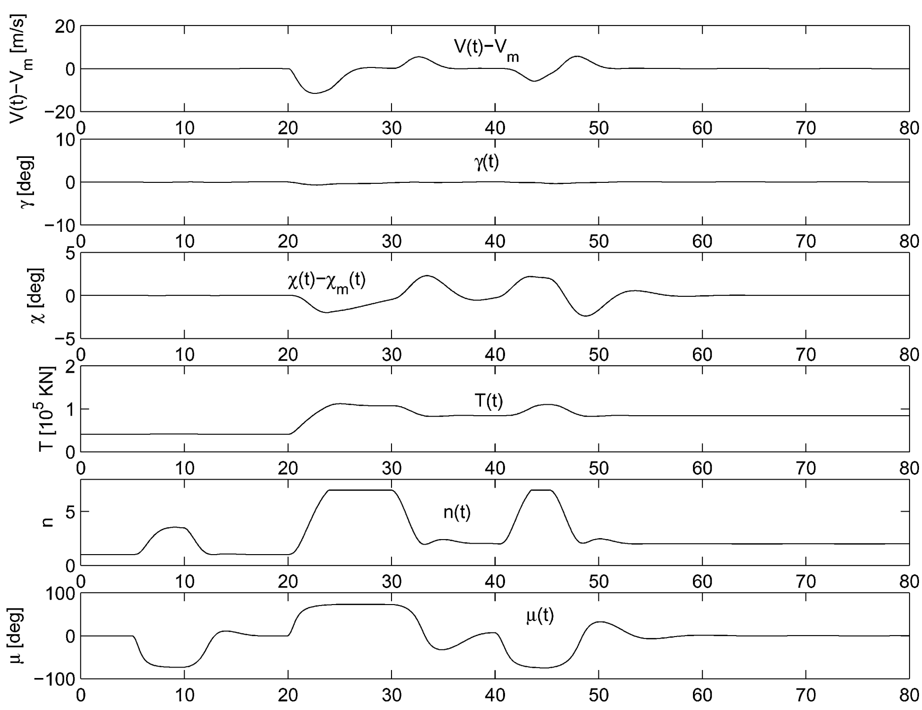

Fig. 2 System response with the baseline CNAP in the case of unknown *p*.

$$\dot{\hat{\boldsymbol{p}}} = \mathrm{Proj}_{[p_{\min}, p_{\max}]}\left\{ -\Gamma G^T(\boldsymbol{x}, \boldsymbol{u}) P \boldsymbol{e}_\eta \right\} \tag{48}$$

where $\boldsymbol{e}_\eta = \boldsymbol{\eta} - \boldsymbol{\eta}_m$, $G(\boldsymbol{x}, \boldsymbol{u}) = -[\Omega^T(\boldsymbol{x}, \boldsymbol{u}),\ J_x^T(\boldsymbol{x}, \boldsymbol{u})]^T$; $P = P^T > 0$ is a solution of $A^T P + P A = -I_{6\times 6}$; and

$$A = \begin{bmatrix} 0 & I_{3\times 3} \\ -I_{3\times 3} & -1.4 I_{3\times 3} \end{bmatrix}$$

$$\Gamma = \mathrm{diag}[0.00000001,\ 0.0001,\ 10000]$$

and $\boldsymbol{\eta}_m$ is a solution of the differential equation $\ddot{\boldsymbol{\eta}}_m + 1.4\dot{\boldsymbol{\eta}}_m + \boldsymbol{\eta}_m = \boldsymbol{r}$, where $r = [0 \quad 0 \quad r_\chi]$.

Simulation 3: Adaptive CNAP for the Case of Unknown Parameters

The response of the system with the adaptive CNAP is shown in Fig. 3. It is seen that the response is substantially improved compared to the case of baseline CNAP. Figure 4 shows the response of the parameter estimates. It is seen that the estimates converge close to true values. However, the convergence is not exact because there is not enough persistent excitation in the system.

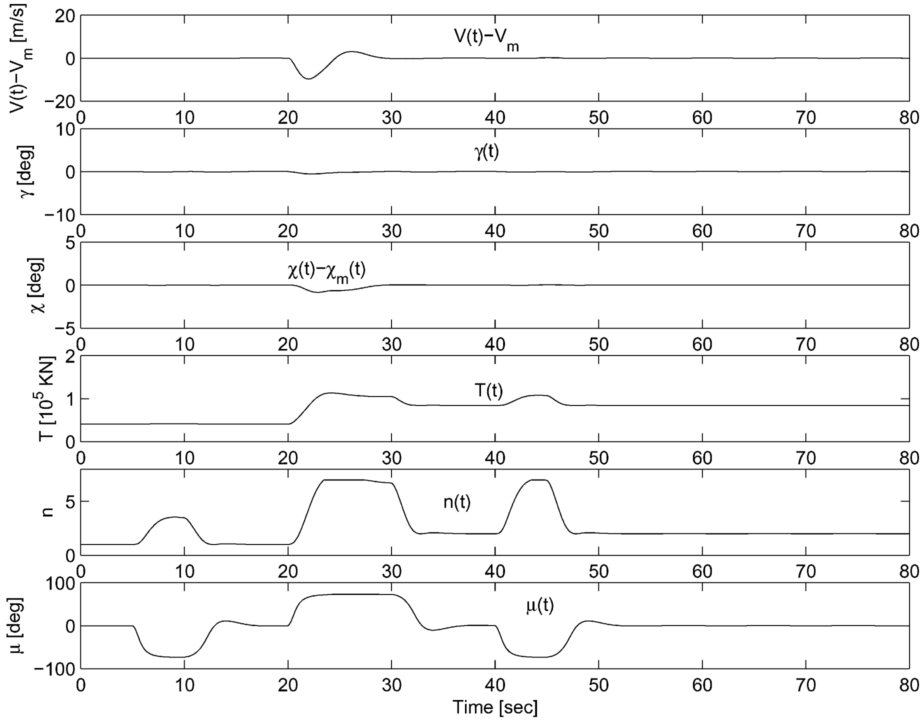

Fig. 3 System response with the adaptive CNAP in the case of unknown p.

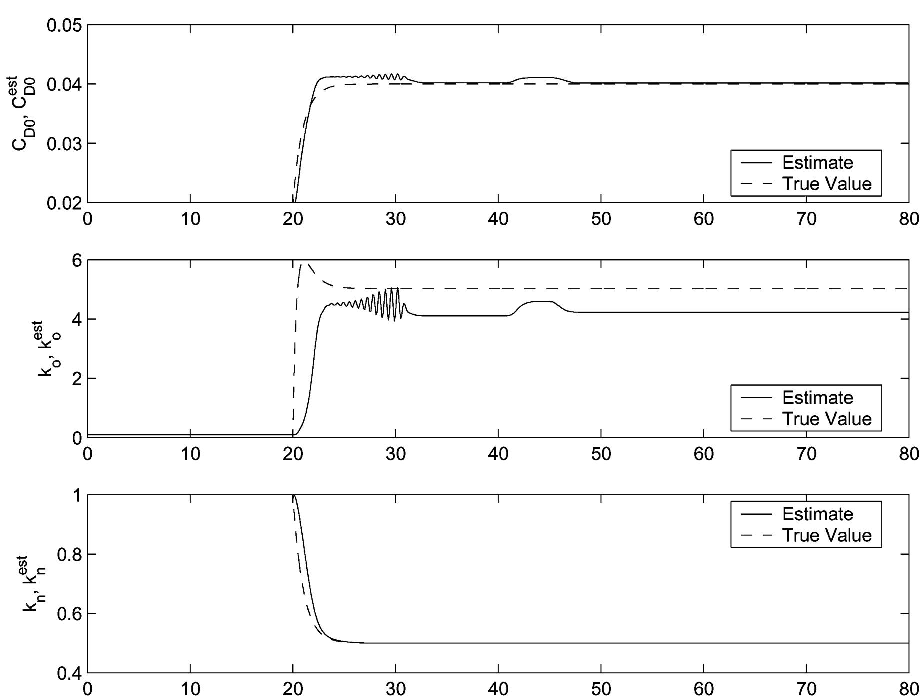

Fig. 4 Response of the parameter estimates with the adaptive CNAP in the case of unknown p.

V. Conclusions

In this paper, adaptive tracking control algorithms are developed for a class of models, which are nonaffine in the control input, encountered in flight control. The essence of the approach is to differentiate the function that is nonlinear in the control input and obtain an increased-order system that is linear in the derivative of the control signal that can be used as a new control variable. A systematic procedure is developed, and related theoretical and practical issues are discussed. The proposed procedure, referred to as CNAP, is developed for several cases of nonaffine models with unknown parameters. The cases include relative degree one models with multiple inputs and outputs, and higher relative degree models with a single input. It is shown that the key aspect in the adaptive control design is the definition of the estimate of the derivative of the system's state, which results in a convenient error model from which the adaptive laws can be written in a straightforward manner. The proposed approach is tested using a 3-DOF simulation of a typical fighter aircraft and is shown to result in substantially improved system response.

The proposed method has an immediate application in the area of flight control where the corresponding nonlinear models are characterized by the control input variables appearing in a non-affine fashion and where model parameters are generally unknown and time-varying. The main limitation of the proposed approach appears to be a highly complex notation and algorithm derivation in the case of high-relative-degree multi-input plants.

Appendix: Adaptive Algorithms with Projection

Properties of such adaptive algorithms will be illustrated in the example of a simple error model:

$$\dot{e} = -\lambda e + \phi\omega(t, x)$$

where $\lambda > 0$; $e = x - x_m$; $x_m : \mathbb{R}^+ \to \mathbb{R}$ is a smooth bounded function; $\omega(t, x)$ is bounded for bounded x and for all time; $\phi = \theta - \theta^*$ denotes the parameter error; θ is an adjustable parameter; and $\theta^* \in [\theta_{\min}, \theta_{\max}]$ is constant. The objective is to adjust $\theta(t)$ within $[\theta_{\min}, \theta_{\max}]$ so that $\lim_{t\to\infty} e(t) = 0$.

Theorem A1: If $\theta(t)$ is adjusted using the adaptive algorithm with projection of the form

$$\dot{\theta} = \mathrm{Proj}_{[\theta_{\min},\theta_{\max}]}\{-\gamma e\omega\}, \qquad \theta(0) \in [\theta_{\min}, \theta_{\max}] \tag{A1}$$

where $\gamma > 0$ and the projection operator is defined as

$$\mathrm{Proj}_{[\theta_{\min},\theta_{\max}]}\{-\gamma e\omega\} = \begin{cases} -\gamma e\omega, & \text{if } \theta(t) = \theta_{\max} \text{ and } e\omega > 0 \\ 0, & \text{if } \theta(t) = \theta_{\max} \text{ and } e\omega \le 0 \\ -\gamma e\omega, & \text{if } \theta_{\min} < \theta(t) < \theta_{\max} \\ 0, & \text{if } \theta(t) = \theta_{\min} \text{ and } e\omega \ge 0 \\ -\gamma e\omega, & \text{if } \theta(t) = \theta_{\min} \text{ and } e\omega < 0 \end{cases}$$

then $\lim_{t\to\infty} e(t) = 0$.

Proof: Let the tentative Lyapunov function for the system be

$$V(e, \phi) = \frac{1}{2}\left(e^2 + \frac{\phi^2}{\gamma}\right) \tag{A2}$$

Its derivative along the motions of the system yields

$$\dot{V}(e, \phi) = -\lambda e^2 + e\phi\omega + \frac{\phi\dot{\phi}}{\gamma}$$

To assure that $\dot{V}$ is negative semidefinite, it is sufficient to show that in all cases

$$\phi\dot{\phi} \le -\gamma e\phi\omega$$

that is, $\phi\dot{\phi} + \gamma e\phi\omega \le 0$. We will further consider each individual case. We note that, since θ^* is constant, $\dot{\theta}(t) \equiv \dot{\phi}(t)$.

1) $\theta(t) = \theta_{\max}$. When $e\omega > 0$, we have that $\dot{\phi} = -\gamma e\omega$ and $\phi\dot{\phi} = -\gamma e\phi\omega$. Because $\phi = \theta - \theta^*$ and $\theta^* \in [\theta_{\min}, \theta_{\max}]$, in this case $\phi = \theta_{\max} - \theta^* \ge 0$. When $e\omega \le 0$, $\dot{\phi} = 0$, and we have $\phi\dot{\phi} + \gamma e\phi\omega = \gamma e\phi\omega \le 0$.

2) $\theta_{\min} < \theta(t) < \theta_{\max}$. In this case, $\phi\dot{\phi} = -\gamma e\phi\omega$.

3) $\theta(t) = \theta_{\min}$. For $e\omega < 0$, we have that $\dot{\phi} = -\gamma e\omega$, and $\phi\dot{\phi} = -\gamma e\phi\omega$. Because, in this case, $\phi = \theta_{\min} - \theta^* \le 0$, for $e\omega \ge 0$, we have $\dot{\phi} = 0$, and $\phi\dot{\phi} + \gamma e\phi\omega = \gamma e\phi\omega \le 0$.

It follows that the adaptive algorithms with projection assure that the condition $\phi\dot{\phi} \le -\gamma e\phi\omega$ is satisfied for all values of arguments. This implies that $\dot{V}(e, \phi) \le -\lambda e^2 \le 0$. Using the arguments from Ref. 16 we can now readily demonstrate that $\lim_{t\to\infty} e(t) = 0$. □

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